Coherent states and the role of the affine group in the quantum mechanics of the Morse potential

2001 J. Phys. A: Math. Gen. 34 3139

(http://iopscience.iop.org/0305-4470/34/14/318)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 3139–3151 www.iop.org/Journals/ja PII: S0305-4470(01)20562-9

Coherent states and the role of the affine group in the quantum mechanics of the Morse potential

B Molnár[1], M G Benedict[1] and J Bertrand[2]

[1] Department of Theoretical Physics, University of Szeged, Tisza L. krt. 84, Szeged, Hungary H-6720

[2] Laboratoire de Physique Theorique de la Matiere Condensee Universite Paris VII, Case 7020, 2 Place Jussieu, 75251 Paris Cedex 05, France

E-mail: benedict@physx.u-szeged.hu and bertrand@ccr.jussieu.fr

Received 23 December 2000

Abstract

The coherent states of the Morse potential that have been obtained earlier from supersymmetric quantum mechanics, are shown to be connected with the representations of the affine group of the real line and some of its extensions. This relation is similar to the one between the Heisenberg–Weyl group and the coherent states of the harmonic oscillator. The states that minimize the uncertainty product of the generators of the affine Lie algebra are shown to contain all the coherent states of the Morse oscillator plus the intelligent states of the Morse Hamiltonians with different shape parameter s. The representations of the central extension of the affine group denoted by G_0 and its further extension $\widetilde{G}_0$ will be shown to define the phase space relevant to the problem by choosing an appropriate orbit of the coadjoint representation of $\widetilde{G}_0$. This allows one to construct a generalized Wigner function on this phase space, which is again essentially in the same relation with the affine group, as the ordinary Wigner function with the Heisenberg–Weyl group.

PACS numbers: 0220S, 0365F, 3420C, 4520

1. Introduction

The Morse Hamiltonian is frequently used in molecular physics, because it is a more realistic model for the description of the vibrations of a diatomic molecule than the harmonic oscillator (HO). In a previous paper [1] the coherent states of this potential were constructed using the technique of supersymmetric quantum mechanics. The aim of the present paper is to show the group theoretical background of this construction and to elucidate the relation between the Morse Hamiltonian and the affine group of the real line and some of its extensions [2]. For the sake of completeness in section 2 we summarize the formalism used in the construction of the eigenstates as well as the coherent states of the Morse Hamiltonian based on supersymmetric quantum mechanics. In section 3 we describe the algebras and groups relevant to the problem, and show that the operators of an irreducible representation of the affine group generate the coherent states from the ground state. A non-trivial extension of the affine group will be shown to have a connection with the minimal (intelligent) states associated with the problem. In section 4 we show the construction of the affine Wigner function based on group theory, and its connection with the ordinary Wigner function.

2. Energy eigenstates and coherent states of the Morse potential

The Morse Hamiltonian to be used here has the following dimensionless form:

$$H(s) = P^2 + \left(s + \tfrac{1}{2} - \exp(-X)\right)^2 . \tag{1}$$

The dimensionless coordinate and momentum obey the usual commutation relation: $[X, P] = \mathrm{i}$, and on the space of functions $\psi(x) \in L^2(-\infty, \infty, \mathrm{d}x)$ X is represented by multiplication by x and P by $-\mathrm{i}\partial_x$. The bound spectrum of this Hamiltonian can be obtained by the technique of supersymmetric quantum mechanics [3,4], a procedure which is based on the factorization of H in terms of generalized ladder operators $A(s)$ and $A^\dagger(s)$:

$$H(s) = A^\dagger(s)A(s) + E_0(s) \tag{2}$$

where

$$\begin{aligned} A(s) &= s - \exp(-X) + \mathrm{i}P \\ A^\dagger(s) &= s - \exp(-X) - \mathrm{i}P \end{aligned} \tag{3}$$

and $E_0(s) = s + \frac{1}{4}$. $A(s)$ and $A^\dagger(s)$ satisfy the following commutation relations:

$$\begin{aligned} &[A(s), A(s)] = 0 \\ &\left[A^\dagger(s), A^\dagger(s)\right] = 0 \\ &\left[A(s), A^\dagger(s)\right] = 2sI - \left(A(s) + A^\dagger(s)\right). \end{aligned} \tag{4}$$

If $s > 0$, then there exists a normalizable ground state $|\Psi_0(s)\rangle$ with energy $E_0(s)$, obeying:

$$A(s)\,|\Psi_0(s)\rangle = 0 \tag{5}$$

i.e. $A(s)$ annihilates the ground state.

The eigenstates can be determined by using the following facts: (a) the supersymmetric partner Hamiltonian, $H^p(s) = A(s)A^\dagger(s) + E_0(s)$ has the same bound energy eigenvalues as H, except for the ground state, and (b) apart from a shift of the parameter s, H^p has the same form as the original H (this is the shape invariance condition) [3,4]: $H^p(s) = H(f(s)) + R(f(s))$, with $f(s) = s - 1$ and $R(s) = 2(s + 1)$. Using these properties together with equation (5) one can determine the energy eigenstates, as well as the eigenvalues of H resulting in

$$\begin{aligned} &|\Psi_n(s)\rangle \propto A^\dagger(s)\cdots A^\dagger(s-n+1)|\Psi_0(s-n)\rangle \\ &E_n(s) = E_0(s) + \sum_{k=1}^{n} R(s-k). \end{aligned} \tag{6}$$

It can be shown that the number of normalizable energy eigenstates is the largest integer that is less than $s + 1$.

In [1] the coherent states of the Morse potential, labelled by a complex number β ($|\beta| < 1$) have been constructed from the ground state in the following way:

$$|\beta\rangle = D(\beta)\,|\Psi_0(s)\rangle$$
$$= \left\{\frac{(1-\beta)}{(1-\beta^*)}\left(1-|\beta|^2\right)\right\}^s \left\{I + \sum_{n=1}^{\infty}\frac{\beta^n}{n!}A^\dagger(s+n-1)\cdots A^\dagger(s)\right\}|\Psi_0(s)\rangle. \tag{7}$$

We note here that the definition of $D(\beta)$ and the phase of the Morse coherent state above differs from the original phase convention used in [1]. According to [1] the coherent state $|\beta\rangle$ would be written as $|\beta\rangle = \mathrm{e}^{-\mathrm{i}\varphi}D(\beta)\,|\Psi_0(s)\rangle$, with $\mathrm{e}^{-\mathrm{i}\varphi} = (|1-\beta|/(1-\beta))^{2s}$. By omitting this irrelevant phase factor the formulae of this paper will have a more simple form. The unitary operator $D(\beta) \equiv D(\widetilde{x}, \widetilde{p})$ generating the coherent state has the alternative exponential form:

$$D(\widetilde{x}, \widetilde{p}) = \exp(-\mathrm{i}\widetilde{p}I)\exp\left(\frac{\widetilde{x}}{2}\left(A^\dagger(s) - A(s)\right)\right)\exp\left(\frac{\mathrm{i}}{2s}\widetilde{p}\left(A(s) + A^\dagger(s)\right)\right) \tag{8}$$

where

$$\widetilde{x} := \ln\left(\mathrm{Re}\,\frac{1+\beta}{1-\beta}\right) \qquad \widetilde{p} := s\frac{\mathrm{Im}[(1+\beta)/(1-\beta)]}{\mathrm{Re}[(1+\beta)/(1-\beta)]}. \tag{9}$$

The parameters $(\widetilde{x}, \widetilde{p})$ are connected with the expectation values of coordinate and momentum in the state $|\beta\rangle$ as $\widetilde{x} = \langle X\rangle - \langle X\rangle_0$ and $\widetilde{p} = \langle P\rangle$, where $\langle X\rangle_0$ is the mean value of x in the ground state. In [1] it has been shown that these states form an overcomplete set for $s > \frac{1}{2}$ with respect to the measure $\delta\beta = (2s-1)/\left(1-|\beta|^2\right)^2 \mathrm{d}\,\mathrm{Re}\,\beta\,\mathrm{d}\,\mathrm{Im}\,\beta$ and that they have the following form in the coordinate representation:

$$\varphi_\beta(y) = \frac{\left(1-|\beta|^2\right)^s}{\sqrt{\Gamma(2s)}|1-\beta|^{2s}}y^s\exp\left(-\frac{y}{2}\frac{1+\beta}{1-\beta}\right) \tag{10}$$

where $y = 2\mathrm{e}^{-x}$. We note that this function has already been found in [5] in another way.

3. The role of the affine group and its extension

In this section we are going to show that the affine group of the real line plays a similar role for the Morse potential as does the Heisenberg–Weyl group for the harmonic oscillator. Let us first briefly clarify the HO case. The operators a, $a^\dagger$ and I used in the theory of the HO form a Lie algebra, usually known as the Heisenberg–Weyl algebra, and the corresponding group is the Heisenberg–Weyl group. This is not a symmetry group of the HO, because the group transformations, in general do not commute with the Hamiltonian. The relevance of this algebra for the HO lies in the following facts: (a) the specific set of generators of the algebra, namely a and $a^\dagger$ are spectrum-generating (ladder) operators, in the sense that they connect eigenstates belonging to neighbouring eigenvalues, and (b) the elements of the irreducible representation of the corresponding group, known as the displacement operators of the HO [6], create the coherent states from the ground state.

We now turn to the case of the Morse potential, and the affine group. The operators in which the Hamiltonian (1) is quadratic are P and $Y = 2\exp(-X)$. Their differential form in the y representation is $P = \mathrm{i}y\partial_y$ and $Y = y$, and they obey the commutation relation:

$$[P, Y] = \mathrm{i}Y. \tag{11}$$

Equation (11) shows that the operators Y and P form a closed Lie algebra. It is known [7], and will be demonstrated briefly below, that this is just the algebra corresponding to the affine group of the real line.

The ladder operators in equation (3) are linear combinations of Y and P plus the identity operator, I. Therefore, we need to extend the algebra spanned by Y and P with the identity operator, which commutes with all the other generators. This standard procedure is known as a central extension in group theory. The operators P, Y and I, or alternatively, A, $A^\dagger$, I, form the basis of an extended Lie algebra, to be denoted here by $\mathcal{G}_0$. The group G_0 corresponding to this algebra has been investigated in [2] for the purpose of signal processing.

In what follows we shall need to use an even larger group, denoted by $\widetilde{G}_0$ which is a further extension of G_0 [2]. We will show in subsection 3.2 that $\widetilde{G}_0$ is closely connected to the minimal states of the strong uncertainty relation of the operators Y and P. Now we are going to characterize these groups and the corresponding algebras in some more detail.

(a) *The affine group*. The elements of the two-parameter affine group (a, b), $a > 0, b \in \mathbb{R}$ act on the points x of the real line as $(a, b)x = ax + b$, and satisfy the group multiplication property: $(a, b)(a', b') = (aa', ab' + b)$.
(b) *The group G_0*. This group is a direct product of the affine group and the real line $\mathbb{R}$ and it can be realized in the following way. G_0 has elements (a, b, c), $a > 0, b, c \in \mathbb{R}$ with the multiplication rule $(a, b, c)(a', b', c') = (aa', ab' + b, c + c')$.
(c) *The group $\widetilde{G}_0$*. This four-parameter group has the following composition rule: $(a, b, c, d)(a', b', c', d') = (aa', ab' + b, c + c' + \rho d' \ln a, d + d')$, where $a > 0, b, c, d \in \mathbb{R}$, and ρ is a fixed real number.

The simplest faithful representation of the group $\widetilde{G}_0$ is given by the four-dimensional matrices

$$g(a, b, c, d) = \begin{pmatrix} a & b & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & c & 1 & \rho \ln a \\ 0 & d & 0 & 1 \end{pmatrix} \tag{12}$$

obeying the required multiplication rule. Disregarding the last row and column, as well as setting $d = 0$ one gets back the group G_0. Disregarding the last two rows and columns, or alternatively setting $c = d = 0$, one arrives at the affine group. The group G_0 can also be recovered as a quotient of $\widetilde{G}_0$ by the subgroup T consisting of the elements $(1, 0, c, 0)$, $c \in \mathbb{R}$. The elements of the quotient $\widetilde{G}_0/T$ are the equivalence classes $[a, b, d] = \{(a, b, c, d), \forall c \in \mathbb{R}\}$. As can be seen from the composition law of $\widetilde{G}_0$ the set of elements $[a, b, d]$ transforms as G_0, and is thus isomorphic to it. The group $\widetilde{G}_0$ is, in fact, constructed as the extension of the group G_0 by the group T [2].

A representation of the generators of the Lie algebra $\widetilde{\mathcal{G}}_0$ corresponding to the group $\widetilde{G}_0$ is obtained by taking the derivatives of $g(a, b, c, d)$ at the unit element. We are going to use here the self-adjoint generators as

$$X_a = \mathrm{i}\, \frac{\partial g}{\partial a}\bigg|_{a=1, b=c=d=0} \tag{13}$$

and similarly for X_b, X_c and X_d. Calculating these derivatives of the matrix above, and evaluating their commutators, one obtains

$$[X_a, X_b] = \mathrm{i} X_b \qquad [X_a, X_d] = \mathrm{i}\rho X_c \qquad [X_b, X_d] = 0 \tag{14}$$

while X_c commutes with all the others. Setting the correspondence:

$$X_a \Rightarrow P \qquad X_b \Rightarrow Y \qquad X_c \Rightarrow I \qquad X_d \Rightarrow \rho \ln Y = Z = -\rho X + (\ln 2)\rho I \tag{15}$$

one arrives at the following commutation relations:

(a)

$$[X_a, X_b] = \mathrm{i}X_b \quad \Rightarrow \quad [P, Y] = \mathrm{i}Y \tag{16}$$

(b)

$$[X_k, X_c] = 0 \quad \Rightarrow \quad [X_k, I] = 0 \qquad k = a, b, c \tag{17}$$

(c)

$$[X_a, X_d] = \mathrm{i}\rho X_c \quad \Rightarrow \quad [P, Z] = \mathrm{i}\rho I \tag{18}$$

$$[X_b, X_d] = 0 \quad \Rightarrow \quad [Y, Z] = 0. \tag{19}$$

Considering only (a), one sees that the commutation relation (11) is really that of the algebra of the affine group, having only these two generators.

(a) and (b) together yield the Lie algebra of the group G_0. In view of equation (4) the ladder operators for the Morse potential can be considered as elements of the extended affine Lie algebra, $\mathcal{G}_0$, and according to equation (6) the eigenstates are generated by the consecutive applications of $A^\dagger$. We note that this Lie algebra corresponds to case III in the Bianchi classification of the three-dimensional Lie algebras [8].

We turn now to the unitary irreducible representations of the above groups on the space of functions $\psi \in L^2(0, \infty; \mathrm{d}y/y)$ with inner product

$$(\psi, \psi') = \int_0^\infty \psi^*(y)\psi'(y)\,\frac{\mathrm{d}y}{y}. \tag{20}$$

The widest group $\widetilde{G}_0$ can be represented unitarily and irreducibly on the function space:

$$U(a, b, c, d)\psi(y) = \exp(-\mathrm{i}(by + c + \rho d \ln y))\,\psi(ay) \qquad (a, b, c, d) \in \widetilde{G}_0 \tag{21}$$

where ρ is a real parameter. Keeping in mind that G_0 and the affine group are subgroups of $\widetilde{G}_0$, one sees that the unitary irreducible representations of G_0 and the affine group can be obtained by setting $d = 0$ and $c = d = 0$, respectively. Alternatively, the whole transformation (21) yields a projective representation for the group $\widetilde{G}_0/T$ consisting of elements $[a, b, d]$.

The generators of the representation (21) of $\widetilde{G}_0$ are:

$$P\psi(y) \equiv \mathrm{i}\frac{\partial U\psi(y)}{\partial a} = \mathrm{i}y\partial_y\psi(y) \tag{22}$$

$$Y\psi(y) \equiv \mathrm{i}\frac{\partial U\psi(y)}{\partial b} = y\psi(y) \tag{23}$$

$$I\psi(y) \equiv \mathrm{i}\frac{\partial U\psi(y)}{\partial c} = \psi(y) \tag{24}$$

$$Z\psi(y) \equiv \mathrm{i}\frac{\partial U\psi(y)}{\partial d} = \rho\,\ln y\,\psi(y) \tag{25}$$

where the derivatives were taken at $a = 1, b = c = d = 0$.

It will be shown in the appendix that the exponentiation of the Lie algebra elements $X = uP + vY + wI + tZ$, yields the following relation between the parameters $u, v, w, t \in \mathbb{R}$ and the group parameters a, b, c, d:

$$\exp\{-\mathrm{i}(uP + vY + wI + tZ)\} = U\left(a = \mathrm{e}^u, b = \frac{v}{u}(\mathrm{e}^u - 1), c = w + \frac{\rho}{2}tu, d = t\right). \tag{26}$$

3.1. The affine group and the Morse coherent states

Let us use the following notation:

$$U(a,b) = U(a,b,0,0) \tag{27}$$

where (a,b) is an element of the affine group. This defines an unitary irreducible representation for the affine group. Since

$$A^\dagger(s) - A(s) = -2\mathrm{i}P \qquad A^\dagger(s) + A(s) = 2sI - Y \tag{28}$$

the unitary operator $D(\widetilde{x},\widetilde{p})$ given by equation (8) can be written as

$$D(\widetilde{x},\widetilde{p}) = \exp(-\mathrm{i}\widetilde{p}I)\exp(-\mathrm{i}\widetilde{x}P)\exp\left(\frac{\mathrm{i}}{2s}\widetilde{p}(2sI - Y)\right) \tag{29}$$

or, since I commutes with P and Y:

$$D(\widetilde{x},\widetilde{p}) = \exp(-\mathrm{i}\widetilde{x}P)\exp\left(-\frac{\mathrm{i}}{2s}\widetilde{p}Y\right). \tag{30}$$

This operator acts on wavefunctions that are defined on the half-line and are square integrable for the measure $\mathrm{d}y/y$. According to (26) and (27) we have

$$\exp(-\mathrm{i}\widetilde{x}P)\exp\left(-\frac{\mathrm{i}}{2s}\widetilde{p}Y\right) = U(\mathrm{e}^{\widetilde{x}},0)\left(1,\frac{\widetilde{p}}{2s}\right). \tag{31}$$

Using

$$U(a,b) = U(1,b)U(a,0) = U(a,0)U\left(1,\frac{b}{a}\right) \tag{32}$$

we obtain that the coherent states are generated from the ground state as

$$|\widetilde{x},\widetilde{p}\rangle = D(\widetilde{x},\widetilde{p})\Psi_0(y) = U(a,b)\Psi_0(y) \tag{33}$$

where the parameters a and b of the affine group are related to $\widetilde{x}$ and $\widetilde{p}$ according to

$$a = \exp(\widetilde{x}) \qquad b = \frac{\widetilde{p}\exp(\widetilde{x})}{2s}. \tag{34}$$

3.2. Minimal states and the group $\widetilde{G}_0$

The coherent states for the Morse potential, obtained in [1] in a different way will be shown here to be the states minimizing the strong uncertainty relation [9, 10] with respect of the non-commuting operators P and Y. The derivation of this relation will now be recalled. Given two Hermitian operators B_1, B_2, the norm of the vector $(B_1 + \mathrm{i}\lambda B_2)\Phi$ is non-negative for any complex number λ:

$$\left\langle\Phi,(B_1 - \mathrm{i}\lambda^* B_2)(B_1 + \mathrm{i}\lambda B_2)\Phi\right\rangle \geqslant 0. \tag{35}$$

Using the notation $\langle B_iB_j\rangle \equiv \langle\Phi, B_iB_j\Phi\rangle$ and setting

$$\lambda = \mathrm{i}\frac{\langle B_2B_1\rangle}{\langle B_2^2\rangle}$$

we obtain by linearity and hermiticity:

$$\langle B_1^2\rangle\langle B_2^2\rangle \geqslant |\langle B_1B_2\rangle|^2 = \tfrac{1}{4}\left(\langle B_1B_2 + B_2B_1\rangle^2 - \langle[B_1,B_2]\rangle^2\right). \tag{36}$$

The latter equality is obtained by splitting $\langle B_1 B_2\rangle$ into its symmetric (real) and antisymmetric (imaginary) parts. Applying this result to

$$B_1 = P - p_0 \qquad B_2 = Y - y_0 \tag{37}$$

with $\langle P\rangle \equiv p_0$ and $\langle Y\rangle \equiv y_0$ leads to the strong uncertainty relation:

$$\sigma_{yy}\sigma_{pp} - \sigma_{py}^2 \geqslant \langle Y\rangle^2/4 \tag{38}$$

where

$$\begin{aligned}\sigma_{yy} &= \langle (Y - y_0)^2\rangle \qquad \sigma_{pp} = \langle (P - p_0)^2\rangle \\ \sigma_{py} &= \frac{\langle PY + YP\rangle}{2} - \langle Y\rangle\langle P\rangle.\end{aligned} \tag{39}$$

According to relations (35) and (37), the minimal uncertainty states for which equation (38) turns into equality obey the equation

$$[(P - p_0) + \mathrm{i}\lambda(Y - y_0)]|\Phi\rangle = 0 \tag{40}$$

where p_0 and y_0 are real and λ is a (generally) complex parameter. In the space of functions $L^2(0, \infty, \mathrm{d}y/y)$ one has to solve the corresponding differential equation:

$$\left[\mathrm{i}y\frac{\partial}{\partial y} - p_0 + \mathrm{i}\lambda(y - y_0)\right]\Phi(y) = 0. \tag{41}$$

The solution is

$$\Phi(y; \lambda, p_0, y_0, \phi) = C\mathrm{e}^{-\mathrm{i}\phi} y^{\lambda y_0 - \mathrm{i}p_0}\mathrm{e}^{-\lambda y} \tag{42}$$

where $C = (2\,\mathrm{Re}\,\lambda)^{(y_0\,\mathrm{Re}\,\lambda)}/\sqrt{\Gamma(y_0 2\,\mathrm{Re}\,\lambda)}$ is fixed by the normalization.

The special case of the Morse coherent states $\varphi_\beta(y)$, defined in equation (10), is obtained for

$$p_0 = y_0\,\mathrm{Im}\,\lambda \qquad \phi = 0.$$

The relation of parameters β, and s to λ and y_0 is written as

$$\beta = \frac{2\lambda - 1}{2\lambda + 1} \qquad s = y_0\,\mathrm{Re}\,\lambda. \tag{43}$$

The normalizability condition, $\mathrm{Re}\,\lambda > 0$ implies $|\beta| < 1$, while the condition $\mathrm{Re}\,\lambda > 1/2y_0$, required for completeness of the set of coherent states, leads to $s > \frac{1}{2}$. Moreover, the Morse coherent states can be obtained by applying the group element $U(a, b)$ to the ground state with the appropriate shape parameter.

On the other hand, the states for which $\sigma_{py} = 0$ are exactly those for which λ is set to be real in equation (40) (see [9, 10]). These are the states which minimize the more customary and weaker form of the uncertainty relation: $\Delta P\,\Delta Y \geqslant \langle Y\rangle/2$. In the family of Morse coherent states, only those with $\mathrm{Im}\,\lambda = 0$ and hence $p_0 = 0$ are minimal in the usual sense. They coincide with the minimal states introduced by Klauder [11] and proved useful as basic analysing wavelets in signal theory [12].

Now let us turn our attention to the problem of how the whole set of minimal states can be generated from the ground state. Considering the representation of $\widetilde{G}_0$ in equation (21) an easy calculation shows that the group elements $U(a, b, c, d)$ generate the minimal states with parameter s which satisfy the constraint $y_0\,\mathrm{Re}\,\lambda = s$:

$$\begin{aligned}&U(a, b, c, d)\,\Psi_0(y, s) = \Phi\left(y; \lambda = \frac{a}{2} + \mathrm{i}b, p_0 = \rho d + \frac{2sb}{a}, y_0 = \frac{2s}{a}, \phi = c\right) \\ &(a, b, c, d) \in \widetilde{G}_0.\end{aligned} \tag{44}$$

Using the analogy with the harmonic oscillator case the above states minimizing the uncertainty relation of (38) and including the coherent states can be regarded as the intelligent states of the Morse potential characterized by the shape parameter s. It is obvious that by changing the shape parameter one obtains disjoint sets of intelligent states which exhaust the entire set of minimal states of the affine algebra.

4. Affine Wigner function

The introduction of Wigner functions in quantum mechanical problems leads to a description in phase space that is equivalent to the operator description, but that is more pertinent in some respects. In particular, the usual coherent and squeezed coherent states of the HO are represented by positive functions and their squeezing is clearly exhibited in the (x, p) phase space.

In the present case, the usual Wigner function is not so well adapted. It is known that the coherent states of (10) will not be represented by a positive Wigner function, (for ordinary Wigner functions of Morse eigenstates, see [13, 14]). More basically, the usual Wigner function is constructed on the Heisenberg group and has a larger covariance by the metaplectic group [15]. As has been shown above, it is the group $\widetilde{G}_0$ which plays a fundamental role in the Morse problem, and hence it seems more desirable to use a phase space representation constructed on this latter group. This will be the affine Wigner function. We will now recall its construction and exhibit some of its properties that are relevant to the present problem.

The natural phase space is spanned here by coordinates p and y. It can be defined as an orbit of group $\widetilde{G}_0$ under its coadjoint representation determined as follows. Consider the adjoint representation of $\widetilde{G}_0$ defined by

$$ad(a, b, c, d)X = U(a, b, c, d)XU^{-1}(a, b, c, d) \tag{45}$$

where $X = x_P P + x_Y Y + x_I I + x_Z Z$ denote the elements of the corresponding Lie algebra. Using the commutation relations (16)–(19) one obtains

$$ad(a, b, c, d)\begin{pmatrix} x_P \\ x_Y \\ x_I \\ x_Z \end{pmatrix} = \begin{bmatrix} 1 & -b & -\rho d & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \rho \ln a & 1 \end{bmatrix} \begin{pmatrix} x_P \\ x_Y \\ x_I \\ x_Z \end{pmatrix}. \tag{46}$$

The coadjoint representation [16, 17] acting on the dual space element $\left(p,\ y,\ \iota,\ z \right)^{\top}$ is given by

$$coad(a, b, c, d)\begin{pmatrix} p \\ y \\ \iota \\ z \end{pmatrix} = \begin{bmatrix} 1 & a^{-1}b & \rho d & 0 \\ 0 & a^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\rho \ln a & 1 \end{bmatrix} \begin{pmatrix} p \\ y \\ \iota \\ z \end{pmatrix}. \tag{47}$$

This representation has four different types of orbits in the dual space of $\widetilde{G}_0$:

$$\begin{aligned} &O^0(\eta_1, \eta_2) = \{(\eta_1, 0, 0, \eta_2)\} \\ &O^H(\xi) = \{(p, 0, \xi, z)\,,\ p, z \in \mathbb{R}\} \\ &O^{\pm}(\xi_1, \xi_2) = \{(p, \pm y, \xi_1, \rho\xi_1 \ln y - \xi_2)\,,\ y > 0,\ p \in \mathbb{R}\} \end{aligned} \tag{48}$$

where the real parameters η_1, η_2, ξ, ξ_1 and ξ_2 are constants characterizing the individual orbits.

Theoretically the above classes of orbits serve four different possibilities to define phase space but not each of them has physical relevance in the present problem. The first class O^0 is singular, because it includes only one-point orbits which are invariant under the action of group $\widetilde{G}_0$ in the coadjoint representation space. It is obvious that these zero-dimensional orbits do not meet the requirements of phase space. The orbits of the second one, O^H give the usual phase space spanned by coordinates (p, x), because the fourth parameter, z originates from the operator $Z = \rho \ln Y = -\rho X + (\ln 2)\rho I$, which is essentially identical to X. Additionally, on these orbits the coadjoint representation above acts as the Heisenberg–Weyl group. This is the consequence of the fact that the subgroup with elements $U(u, v = 0, w, t)$ is identical to the Heisenberg–Weyl group. We have shown, however, that the operators P and Y are the relevant operators in the Morse problem and it is obvious that by choosing the class O^H, the affine subgroup of $\widetilde{G}_0$ would be lost. Hence it is seen that the appropriate choice of the phase space is an orbit from the class O^+ or O^- parametrized by (p, y). Among them only the class O^+ with positive-valued coordinate y is well adapted to the current problem. The action of the group $\widetilde{G}_0$ on the phase space corresponding to the orbit $O^+(\xi_1, \xi_2)$ can be written as

$$\begin{pmatrix} p' \\ y' \end{pmatrix} = \begin{pmatrix} p + a^{-1}by + \rho\xi_1 d \\ a^{-1}y \end{pmatrix}. \tag{49}$$

This gives the action of $\widetilde{G}_0$ on the phase space parametrized by coordinates (p, y) and consequently on any function $F(p, y)$:

$$(a, b, c, d) : F(p, y) \longrightarrow F'(p, y) \equiv F(p - by - \rho\xi_1 d, ay). \tag{50}$$

In the following, we will set $\xi_1 = 1$ as this does not restrict generality.

Now the affine Wigner function is defined as a real quadratic functional of the wavefunction written as

$$W(p, y; \psi) = \int_{\mathbb{R}^+\times\mathbb{R}^+} K(p, y; y_1, y_2)\psi(y_1)\psi^*(y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2 \tag{51}$$

where the kernel K is such that

$$K^*(p, y; y_1, y_2) = K(p, y; y_2, y_1). \tag{52}$$

Now, the correspondence $\psi(y) \to W(p, y; \psi)$ has to fulfil the following requirements.

1. *Covariance by the group* $\widetilde{G}_0$. When the wavefunction $\psi(y)$ is transformed by the representation $U(a, b, c, d)$ according to (21), the corresponding phase space function $W(p, y; \psi)$ undergoes a pointlike transformation of the form (50). In other words, the following diagram is commutative:

$$\begin{array}{ccc} \psi(y) & \longrightarrow & U(a,b,c,d)\psi(y) = \exp(-\mathrm{i}(by + c + \rho d \ln y))\,\psi(ay) \\ \downarrow & & \downarrow \\ W(p,y) & \longrightarrow & W'(p,y) \equiv W(p - by - \rho d, ay). \end{array} \tag{53}$$

   Writing explicitly this condition with the functional $W(p, y; \psi)$ defined in (51) leads necessarily to a phase space function of the form:

$$W(p, y; \psi) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}vp}\,\psi\left(\frac{yv\mathrm{e}^{v/2}}{2\sinh(v/2)}\right)\psi^*\left(\frac{yv\mathrm{e}^{-v/2}}{2\sinh(v/2)}\right)\mu(v)\,\mathrm{d}v \tag{54}$$

   where the function $\mu(v)$ is such that

$$\mu^*(v) = \mu(-v) \tag{55}$$

   and otherwise arbitrary.

2. *Unitarity.* This is also called Moyal property in the case of the usual Wigner function. Here it is written as

$$\int_{\mathbb{R}\times\mathbb{R}^+} W(p, y; \psi_1) W(p, y; \psi_2)\, \mathrm{d}p\, (\mathrm{d}y/y) = |(\psi_1, \psi_2)|^2 \tag{56}$$

with the scalar product of wavefunctions being given by equation (20). The constraint (56) leads to the determination of $\mu(v)$: $\mu(v) = \frac{1}{2\pi}$.

Finally, the affine Wigner function is given by

$$W(p, y; \psi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}vp}\, \psi\left(\frac{yv\mathrm{e}^{v/2}}{2\sinh(v/2)}\right) \psi^*\left(\frac{yv\mathrm{e}^{-v/2}}{2\sinh(v/2)}\right) \mathrm{d}v. \tag{57}$$

This function is not positive everywhere for an arbitrary state, and is thus only a pseudo-distribution. However, it provides a phase space interpretation for the properties relative to the minimal states and for the action of the group $\widetilde{G}_0$.

Due to the non-commutativity of the affine algebra, there is no state having a phase space representation $W(p, y)$ exactly localized in one point. The optimal concentration of the affine Wigner function will occur for the minimal states. Consider first the Morse ground state, which is identical to the coherent state with $\beta = 0$:

$$\Psi_0(y) = \frac{1}{\sqrt{\Gamma(2s)}} y^s \mathrm{e}^{-(y/2)}. \tag{58}$$

Substitution into expression (57) gives

$$W(p, y; \Psi_0(s)) = \frac{y^{2s}}{2\pi\Gamma(2s)} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}vp} \left(\frac{v}{2\sinh(v/2)}\right)^{2s} \mathrm{e}^{-y(v/2)\coth(v/2)}\, \mathrm{d}v$$
$$\equiv W_0(p, y; s). \tag{59}$$

This expression can be shown to be positive, as the Fourier transform of a function of positive type [18], and it is localized around the point $p = 0$, $y = 2s$. The use of the covariance property (53) shows that the phase space representation of a general minimal state, as defined by equation (44), is obtained from W_0 by a point transformation and is thus positive as well. In particular, the affine Wigner representation of any Morse coherent state (10) written as

$$\varphi_\beta(y) = U(a, b)\Psi_0(y) = \frac{a^s}{\sqrt{\Gamma(2s)}} y^s\, \mathrm{e}^{-(a/2+\mathrm{i}b)y} \qquad a/2 + \mathrm{i}b = \frac{1}{2}\frac{1+\beta}{1-\beta} \tag{60}$$

is equal to

$$W(p, y; \varphi_\beta) = W_0(p - by, ay; s). \tag{61}$$

Thus, for a given s, there is a Morse coherent state attached to any point of the phase space and the extension of its (positive) affine Wigner function depends only on that point.

More generally, any intelligent state (42) is represented by

$$W(p, y; \Phi(\lambda, p_0, y_0, \phi)) = W_0(p - p_0 - (y - y_0)\,\mathrm{Im}\,\lambda, 2\,\mathrm{Re}\,\lambda\, y; y_0\,\mathrm{Re}\,\lambda). \tag{62}$$

The parameter λ, which appeared in equation (40), can thus be interpreted as a factor determining the shape of the function $W(p, y)$. In the limit $\lambda = 0$, the functions $\Phi(y)$ collapse into eigenstates of the operator P:

$$\psi_P(y) = y^{-\mathrm{i}p_0} \tag{63}$$

and sharp localization is obtained in phase space:

$$W(p, y; \psi_P) = \delta(p - p_0). \tag{64}$$

The group $\widetilde{G}_0$ has been seen to perform point transformations in the affine Wigner function. In fact, it is the largest group with this property which can be represented unitarily on the wavefunction. It thus plays a role analogous as does the metaplectic group [15] in the harmonic oscillator case.

To make a comparison of the present phase space representation (57) with what would be obtained from Wigner's, it is useful to go back to the original variable x and to define

$$\widetilde{\psi}(x) \equiv \psi(2\mathrm{e}^{-x}) \qquad \widetilde{W}(p, x; \widetilde{\psi}) \equiv W(p, 2\mathrm{e}^{-x}; \psi). \tag{65}$$

Expression (57) leads to

$$\widetilde{W}(p, x; \widetilde{\psi}) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}vp} \widetilde{\psi}\left(x - \frac{v}{2} - \ln\frac{v}{2\sinh(v/2)}\right) \widetilde{\psi}^*\left(x + \frac{v}{2} - \ln\frac{v}{2\sinh(v/2)}\right) \mathrm{d}v. \tag{66}$$

It can be seen that, in the vicinity of $v = 0$, the argument of ψ (respectively ψ^*) is approximated by $x - v/2$ (respectively $x + v/2$). Hence the function $\widetilde{W}(p, x; \widetilde{\psi})$ will be close to the original Wigner function only for states $\psi(x)$ which have support over a narrow interval. In particular, the usual Wigner function cannot localize exactly states of the form (63).

5. Conclusions and final remarks

In this work we have presented the role of the affine group and its extensions G_0 and $\widetilde{G}_0$ in the quantum mechanics of the Morse potential. We have shown that the essential operators of the Morse problem, namely the momentum P and the exponentially scaled position Y, form the Lie algebra of the affine group of the real line. Moreover, we have proven that the algebra spanned by the spectrum-generating SUSY ladder operators and the identity is isomorphic to the Lie algebra of the centrally extended affine group G_0. As a consequence of these facts, we have established a connection between the unitary irreducible representations of the groups above and certain sets of states relevant to the Morse Hamiltonian.

First, we have shown that the overcomplete set of the Morse coherent states constructed in [1] can also be generated from the ground state by the elements of the unitary irreducible representation of the affine group on the Hilbert space. Therefore, these states are the coherent states of the affine group in Perelomov's sense [19,20] too. Furthermore, we have investigated the strong uncertainty relation related to the affine algebra generators P and Y and the states which minimize this inequality. It has turned out that the set of these minimal states can be decomposed into disjoint subsets of intelligent states belonging to different Morse potentials characterized by different shape parameters s. We would like to note here that in the case of the harmonic oscillator the set of intelligent states is identical to the whole set of minimal states of the uncertainty relation. This follows from the fact that the frequency playing the shape parameter for the harmonic oscillator [3] does not change by constructing the SUSY partner Hamiltonian. This is in contrast with the case of Morse partner Hamiltonians where the shape parameter s is shifted. We have also shown that each of these sets of intelligent states are generated from the corresponding ground state with the appropriate shape parameter s by the elements of the unitary irreducible representation of group $\widetilde{G}_0$. Due to these arguments this latter group plays the analogous role for the Morse potential as the metaplectic group does for the oscillator.

In view of these observations above it seemed plausible to introduce the phase space for the Morse potential through the group $\widetilde{G}_0$. Following the method of Kirillov we have considered the coadjoint orbits of $\widetilde{G}_0$ and have given the phase space as a two-dimensional manifold with coordinates p and y. Putting natural requirements on the functions on the phase space we have arrived at a phase space distribution function obeying the covariance property under $\widetilde{G}_0$ and the Moyal property. This phase space distribution function called the affine Wigner function is the same that as originally introduced in the context of signal processing theory [2].

Acknowledgments

This work was initiated while two of the authors (MGB and JB) were participating in the workshop: Optical Systems and their Wigner functions at the Centro Internacional de Ciencias, Cuernavaca, Mexico. We are grateful for financial support of that institute and especially to K B Wolf for his hospitality, and for useful discussions. Two of the authors (BM and MGB) are also grateful to L Fehér for clarifying remarks. The work has been supported by the National Research Foundation of Hungary (OTKA) under contracts nos T22281 and T32920.

Appendix. Baker–Campbell–Hausdorff-type relation for $\widetilde{G}_0$

In order to obtain the Baker–Campbell–Hausdorff-type relation for $\widetilde{G}_0$ given in equation (26) let us recall the four-dimensional matrix representation given by equation (12). We will prove that the identifications of parameters

$$a = \mathrm{e}^u \qquad b = \frac{v}{u}(\mathrm{e}^u - 1) \qquad c = w + \frac{\rho}{2}ut \qquad d = t \tag{A1}$$

yield

$$\exp\{-\mathrm{i}(uX_a + vX_b + wX_c + tX_d)\} = g(a, b, c, d) \qquad (a, b, c, d) \in \widetilde{G}_0. \tag{A2}$$

The Lie algebra generators X_a, X_b, X_c and X_d are defined by equation (13) and the parameters u, v, w and t are arbitrary real numbers.

To prove the identity above let us first set $d = 0$ and calculate the left-hand side of equation (A2). A straightforward calculation gives

$$\exp\{-\mathrm{i}(uX_a + vX_b + wX_c)\} = \begin{pmatrix} \mathrm{e}^u & v\dfrac{\mathrm{e}^u - 1}{u} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & w & 1 & \rho u \\ 0 & 0 & 0 & 1 \end{pmatrix} = g\left(\mathrm{e}^u, v\frac{\mathrm{e}^u - 1}{u}, w, 0\right). \tag{A3}$$

In the case of arbitrary d we can use the commutators (16)–(19). With the notation $B \equiv -\mathrm{i}tX_d$ and $C \equiv -\mathrm{i}(uX_a + vX_b + wX_c)$ one has $[B, C] = -\mathrm{i}ut\rho X_d$ and $[[B, C], B] = [[B, C], C] = 0$. Then $\mathrm{e}^{\{B+C\}} = \mathrm{e}^{-\frac{1}{2}[B,C]}\mathrm{e}^A\mathrm{e}^B$ and one can write

$$\exp\{-\mathrm{i}(uX_a + vX_b + wX_c + tX_d)\} = \exp(-\mathrm{i}tX_d)\exp\big\{-\mathrm{i}\big(uX_a + vX_b + \big(w + \tfrac{1}{2}ut\rho\big)X_c\big)\big\}. \tag{A4}$$

Using equation (A3) and (A4):

$$\exp\{-\mathrm{i}(uX_a + vX_b + wX_c + tX_d)\} = g(1, 0, 0, d)g(a, b, c, 0) \tag{A5}$$

where $a = \mathrm{e}^u$, $b = \frac{v}{u}(\mathrm{e}^u - 1)$, $c = w + \frac{\rho}{2}ut$ and $d = t$. Finally, taking into account that $(a, b, c, d) = (1, 0, 0, d)(a, b, c, 0)$ one arrives at equation (A2).

References

[1] Benedict M G and Molnár B 1999 *Phys. Rev.* A **60** R1737
[2] Bertrand J and Bertrand P 1992 *J. Math. Phys.* **33** 2515
[3] Dutt R, Khare A and Sukhatme U P 1988 *Am. J. Phys.* **56** 163
Cooper F, Khare A and Sukhatme U P 1995 *Phys. Rep.* **251** 268
[4] Gendenshtein L E 1983 *Pis. Zh. Eksp Teor. Fiz.* **38** 299 (Engl. transl. 1983 *JETP Lett.* **38** 356)
[5] Nieto M M and Simmons L M 1979 *Phys. Rev.* D **20** 1321
Nieto M M and Simmons L M *Phys. Rev.* D **20** 1342
[6] Glauber R J 1963 *Phys. Rev.* **130** 2529 *Phys. Rev.* **131** 2766
[7] Aslaksen E W and Klauder J R 1968 *J. Math. Phys.* **9** 206
Aslaksen E W and Klauder J R 1969 *J. Math. Phys.* **10** 2267
[8] Dubrovin B A, Novikov S P and Fomenko T 1985 *Modern Geometry* vol 1 (Berlin: Springer)
[9] Merzbacher E 1961 *Quantum Mechanics* (New York: Wiley)
[10] Golebiewski A and Witko M 1988 *Phys. Rev.* A **38** 2715
[11] Klauder J R 1980 *Functional Integration: Theory and Applications* ed J P Antoine and E Tirapegui (New York: Plenum) p 110
[12] Bertrand J and Bertrand P 1996 *IEEE Trans. Geosci. Remote Sensing* **34** 1144
[13] Lee H W and Scully M O 1982 *J. Chem. Phys.* **77** 4605
[14] Frank A, Rivera A L and Wolf K B 2000 *Phys. Rev.* A **61** 054102
[15] Garcia-Bullé M, Lassener W and Wolf K B 1986 *J. Math. Phys.* **27** 29
[16] Kirillov A 1976 *Elements of the Theory of Representations* (Berlin: Springer)
[17] Vilenkin N J and Klimyk A U 1991 *Representation of Lie Groups and Special Functions* vol 1 (Dordrecht: Kluwer)
[18] Flandrin P 1998 *J. Math. Phys.* **39** 4016
[19] Perelomov A M 1986 *Generalized Coherent States and Their Applications* (Berlin: Springer)
[20] Ali T S, Antoine J P and Gazeau J P 2000 *Coherent States, Wavelets and their Generalizations* (Berlin: Springer)